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Structural stability and robustness to bounded rationality for non-compact cases

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Abstract We study the model M consisting of "general games" with noncompact action space, together with an associated abstract rationality function. We prove that M is structurally stable and robust to ϵ-equilibria for "almost all" parameters. As applications, we investigate structural stability and robustness to bounded rationality for noncooperative games, multi-objective optimizations and fixed point problems satisfying existence and some continuity conditions. Specifically, we introduce concrete rationality functions for such three kinds of problems with both payoffs and strategy sets, objective functions and domain spaces, and correspondence and domain spaces as parameters, respectively, and show the generic structural stability and robustness to bounded rationality for the corresponding model Ms.

Keywords Structural stability · Robustness · Bounded rationality · Nash equilibrium · Multiobjective optimization · Fixed point

1 Introduction and preliminaries

One of the standard interpretations of noncooperative game theory is the full rationality of players. As is well known, perfect rationality that underlies most economic models is far too strict and bounded rationality is more reasonable as a basis for economic analysis, see [6] and [7] and references therein. Anderlini and Canning [1] established an abstract framework, model M, a parameterized class of "general games" together with an associated abstract rationality function, and defined ϵ-equilibria, equilibria and robustness to ϵ-equilibria. Then they introduced the notion of structural stability: a model is structurally stable if the equilibrium

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set (under full rationality) varies continuously with changes in the parameter values. They proved that, under some conditions, model M is robust to ϵ-equilibria if and only if it is structurally stable. Following [1], under some weaker assumptions than those in [1], Yu and Yu [8,9] showed that the model M is structurally stable and robust to ϵ-equilibria for "almost all" parameter values.

A model M is a quadruple $\{\Lambda, X, F, R\}$ with a parameter space Λ, an action space X, a feasibility correspondence $F: \Lambda \times X \to 2^X$ which defines a further correspondence $f: \Lambda \to 2^X$, $f(\lambda) = \{x \in X: x \in F(\lambda, x)\}$, and a rationality function $R: \text{Graph}(f) \to \mathbb{R}_+$, with $R(\lambda, x) = 0$ corresponding to the full rationality, where $\text{Graph}(f) = \{(\lambda, x) \in \Lambda \times X: x \in f(\lambda)\}$. For any $\lambda \in \Lambda$ and any $\epsilon \geq 0$, the set of ϵ-equilibria at λ is defined as

$$E(\lambda, \epsilon) = \{x \in f(\lambda) : R(\lambda, x) \leq \epsilon\}.$$

Particularly, the set of equilibria at λ is defined as

$$E(\lambda) = E(\lambda, 0) = \{x \in f(\lambda) : R(\lambda, x) = 0\}.$$

The main assumptions here are:

(1) (Λ, ρ) is a complete metric space, (X, d) is a metric space (may be noncompact);
(2) $f: \Lambda \to 2^X$ is upper semicontinuous, and for any $\lambda \in \Lambda, f(\lambda)$ is nonempty and compact;
(3) $R: \text{Graph}(f) \to \mathbb{R}_+$ is lower semicontinuous; and
(4) for any $\lambda \in \Lambda$, $E(\lambda) \neq \emptyset$.

Clearly, under these assumptions, $E(\lambda, \epsilon)$ is compact for any $(\lambda, \epsilon) \in \Lambda \times \mathbb{R}_+$.

As the same in [8,9], the model M is robust to ϵ-equilibria at $\lambda \in \Lambda$ if for any $\delta > 0$, there exists an $\bar{\epsilon} > 0$ such that for any ϵ with $0 < \epsilon < \bar{\epsilon}$ and any $\lambda' \in \Lambda$ with $\rho(\lambda', \lambda) < \bar{\epsilon}, h(E(\lambda', \epsilon), E(\lambda')) < \delta$, where h is the Hausdorff distance on X. The model M is structurally stable at $\lambda \in \Lambda$ if the equilibrium correspondence $E: \Lambda \to 2^X$ is continuous at λ.

Clearly, the assumptions above are weaker than those in [8,9]. But under the weaker conditions, we will show that theorems in [8,9] still hold, and we will give some applications.

Now we recall some notions of continuity of correspondence. Let X and Y be two metric spaces, $K(X)$ be the set of all nonempty compact subsets of X and $F: Y \to K(X)$ be a correspondence. F is said to be upper semicontinuous at $y \in Y$ if for any open set U in X with $U \supset F(y)$, there is an open neighborhood $O(y)$ of y such that $U \supset F(y')$ for each $y' \in O(y)$; F is said to be lower semicontinuous at y if for any open set U with $U \cap F(y) \neq \emptyset$, there is an open neighborhood $O(y)$ of y such that $U \cap F(y') \neq \emptyset$ for each $y' \in O(y)$; F is continuous at y if it is both upper and lower semicontinuous at y. F is said to be upper semicontinuous (lower semicontinuous, or continuous) on Y if it is upper semicontinuous (lower semicontinuous, or continuous) at every $y \in Y$.

Lemma 1.1 *Let X be a metric space, $\{A_n\}$ be a sequence of nonempty subsets of X and A be a nonempty compact subset of X. Suppose that for any open set O in X with $O \supset A$, there is a positive integer N such that for any $n \geq N$, $O \supset A_n$. Then for any sequence $\{x_n\}$ with $x_n \in A_n, n = 1, 2, \ldots,$ there is a cluster point x^* of $\{x_n\}$ such that $x^* \in A$.*

Proof By way of contradiction, suppose that the opposite conclusion holds. Then for any $x \in A$, x is not a cluster point of $\{x_n\}$, that is, there are an open neighborhood $O(x)$ and a positive integer $n(x)$ such that $x_n \notin O(x)$ for any $n \geq n(x)$. Since $A \subset \cup_{x \in A} O(x)$ and A is compact, there are $x^1, x^2, \ldots, x^p \in A$ such that $A \subset \cup_{i=1}^p O(x^i)$. Let $n_0 = \max\{n(x^1), \ldots, n(x^p)\}$, then we have $x_n \notin \cup_{i=1}^p O(x^i)$ for all $n \geq n_0$. Since $\cup_{i=1}^p O(x^i)$ is an open set in X and

$\cup_{i=1}^{p} O(x^i) \supset A$, then there is $n_1 \geq n_0$ such that $\cup_{i=1}^{p} O(x^i) \supset A_n$ for all $n \geq n_1$. Finally we obtain $x_{n_1} \notin \cup_{i=1}^{p} O(x^i)$ and $x_{n_1} \in A_{n_1} \subset \cup_{i=1}^{p} O(x^i)$, a contradiction. □

Lemma 1.2 *Let Λ and X be two metric spaces, $\lambda \in \Lambda$. Suppose that $f\colon \Lambda \to K(X)$ is upper semicontinuous at λ. Then for any $\lambda_n \to \lambda$ and any $x_n \in f(\lambda_n)$, there is a cluster point x^* of $\{x_n\}$ such that $x^* \in f(\lambda)$.*

Proof Since f is upper semicontinuous at λ and $\lambda_n \to \lambda$, for any open set O in X with $O \supset f(\lambda)$, there is a positive integer N such that $O \supset f(\lambda_n)$ for all $n \geq N$. Then it follows the conclusion from Lemma 1.1. □

The following lemma is due to Lemma 2.5 of [10].

Lemma 1.3 *Let X and Y be two metric spaces, $A_m, A \in K(X)$, $y_m, y \in Y$ and g^m, g be continuous functions defined on $X \times Y, m = 1, 2, \ldots$. If $h(A_m, A) \to 0$, where h is the Hausdorff distance defined on X, $y_m \to y$ and $\sup_{(x,y)\in X\times Y} |g^m(x, y) - g(x, y)| \to 0$, then*

$$\max_{w\in A_m} g^m(w, y_m) \to \max_{w\in A} g(w, y).$$

The following lemma 1.4 is due to Theorem 2 of Fort [3], also see Lemma 2.1 of [10].

Lemma 1.4 *Let Y be a complete metric space, X be a metric space and $F\colon Y \to K(X)$ be an upper semicontinuous correspondence. Then there exists a dense G_δ set Q of Y such that F is continuous at every $y \in Q$.*

2 Main results

Theorem 2.1 *Let (Λ, ρ) be a complete metric space, (X, d) be a metric space, $f\colon \Lambda \to K(X)$ be an upper semicontinuous correspondence, $R\colon \mathrm{Graph}(f) \to \mathbb{R}_+$ be a lower semicontinuous function and $E(\lambda) \neq \emptyset$ for any $\lambda \in \Lambda$. Then*

(1) the equilibrium correspondence $E\colon \Lambda \to K(X)$ is upper semicontinuous; and
(2) there is a dense G_δ subset Q of Λ such that for any $\lambda \in Q$, M is structurally stable.

Proof (1) First, for any $\lambda \in \Lambda$, $E(\lambda) = \{x \in f(\lambda) : R(\lambda, x) = 0\} = f(\lambda) \cap \{x \in X : R(\lambda, x) \leq 0\}$. The lower semicontinuity of R implies that $\{x \in X : R(\lambda, x) \leq 0\}$ is closed and hence $E(\lambda)$ is compact since $f(\lambda)$ is compact.

Next we show that E is upper semicontinuous at λ. By way of contradiction, suppose that there is an open set O of X with $O \supset E(\lambda)$ such that there are a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda$ and a sequence $\{x_n\}$ with $x_n \in E(\lambda_n)$, $n = 1, 2, 3, \ldots$, but $x_n \notin O$. Note that $x_n \in f(\lambda_n)$ since $x_n \in E(\lambda_n)$. Since f is upper semicontinuous at λ and $f(\lambda)$ is compact, by Lemma 1.2, there is a cluster point x of $\{x_n\}$ with $x \in f(\lambda)$, that is, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying $x_{n_k} \to x$. Note that $x_{n_k} \in E(x_{n_k})$ and thus $R(\lambda_{n_k}, x_{n_k}) = 0$. Since R is lower semicontinuous at (λ, x), we have $R(\lambda, x) \leq \liminf_{n_k\to\infty} R(\lambda_{n_k}, x_{n_k}) = 0$. Hence $x \in E(\lambda) \subset O$, which is a contradiction since $x_{n_k} \to x$ and O is open but $x_{n_k} \notin O$ for all n_k. This proves that the equilibrium correspondence $E\colon \Lambda \to K(X)$ is upper semicontinuous.

(2) By Lemma 1.4, there exists a dense G_δ subset Q of Λ such that the equilibrium correspondence $E\colon \Lambda \to K(X)$ is continuous at every $\lambda \in Q$. Hence, M is structurally stable at every $\lambda \in Q$. □

Theorem 2.2 *Under the assumptions of Theorem 2.1, if M is structurally stable at $\lambda \in \Lambda$, then M is robust to ϵ-equilibria at $\lambda \in \Lambda$.*

Proof Also by way of contradiction, suppose that M is not robust to ϵ-equilibria at some $\lambda \in \Lambda$. Then there are a $\delta_0 > 0$, a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0$ and $\epsilon_n \to 0$ and a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda$ such that

$$h(E(\lambda_n, \epsilon_n), E(\lambda_n)) \geq \delta_0.$$

Since $E(\lambda_n) \subset E(\lambda_n, \epsilon_n)$, we can select $x_n \in E(\lambda_n, \epsilon_n)$ such that

$$\min_{w \in E(\lambda_n)} d(x_n, w) > \frac{\delta_0}{2}.$$

Furthermore, since $x_n \in E(\lambda_n, \epsilon_n)$, then $x_n \in f(\lambda_n)$ and $R(\lambda_n, x_n) \leq \varepsilon_n$. Since $f\colon \Lambda \to K(X)$ is upper semicontinuous at λ and $f(\lambda)$ is compact, by Lemma 1.2, $\lambda_n \to \lambda$ and $x_n \in f(\lambda_n)$ imply that there is a cluster point x of $\{x_n\}$ such that $x \in f(\lambda)$, i.e., $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ satisfying $x_{n_k} \to x$. By lower semicontinuity of R at (λ, x), we have $R(\lambda, x) \leq \liminf_{n_k \to \infty} R(\lambda_{n_k}, x_{n_k}) = 0$. Hence $x \in E(\lambda)$.

Since M is structurally stable at $\lambda \in \Lambda$, the correspondence E: $\Lambda \to K(X)$ is continuous at λ, i.e., $h(E(\lambda_{n_k}), E(\lambda)) \to 0$. By Lemma 1.3, we get

$$\min_{w \in E(\lambda)} d(x, w) \geq \frac{\delta_0}{2},$$

which contradicts that $x \in E(\lambda)$. Hence M must be robust to ε-equilibria at $\lambda \in \Lambda$. □

It follows from the same argument as the proof of Theorems 3.3 and 3.4 in [8] that

Theorem 2.3 *Under the assumptions of Theorem 2.1, there exists a dense G_δ subset Q of Λ such that $\forall \lambda \in Q, \forall \lambda_n \to \lambda, \forall \epsilon_n \to 0$, we have*

$$h(E(\lambda_n, \epsilon_n), E(\lambda)) \to 0.$$

Theorem 2.4 *Under the assumptions of Theorem 2.1, if $\lambda \in \Lambda$ is such that $E(\lambda)$ is a singleton set, then M is structurally stable and robust to ϵ-equilibria at $\lambda \in \Lambda$.*

3 Applications

3.1 n-person noncooperative games

Let $I = \{1, \ldots, n\}$ be the set of players. For each $i \in I$, X_i, a metric space, is the strategy set and u_i: $X = \prod_{i=1}^n X_i \to \mathbb{R}$ is the payoff function of i-th player, respectively. For each $i \in I$, denote $\hat{i} = I \setminus \{i\}$.

$\Lambda = \{\lambda = (u_1, \ldots, u_n; S_1, \ldots, S_n) : \forall i \in I, u_i$ is continuous on X and $\sup_{x \in X} \sum_{i=1}^n |u_i(x)| < +\infty$, S_i is a nonempty compact subset of X_i and $\exists x \in S = \prod_{i=1}^n S_i$ such that $\forall i \in I, u_i(x_i, x_{\hat{i}}) = \max_{w_i \in S_i} u_i(w_i, x_{\hat{i}})\}$.

For any $\lambda_1 = (u_{11}, \ldots, u_{1n}; S_{11}, \ldots, S_{1n}), \lambda_2 = (u_{21}, \ldots, u_{2n}; S_{21}, \ldots, S_{2n}) \in \Lambda$, define

$$\rho(\lambda_1, \lambda_2) = \sup_{x \in X} \sum_{i=1}^n |u_{1i}(x) - u_{2i}(x)| + \sum_{i=1}^n h_i(S_{1i}, S_{2i})$$

where $\forall i \in I$, h_i is the Hausdorff metric on X_i.

It is easy to check that ρ is a metric. Next we show that

Lemma 3.1 (Λ, ρ) *is a complete metric space.*

Proof Let $\{\lambda_m\}$ be any Cauchy sequence in Λ. Then for any $\varepsilon > 0$, there is a positive integer $P(\varepsilon)$ such that for any $m, p \geq P(\varepsilon)$,

$$\rho(\lambda_m, \lambda_p) = \sup_{x \in X} \sum_{i=1}^{n} |u_{mi}(x) - u_{pi}(x)| + \sum_{i=1}^{n} h_i(S_{mi}, S_{pi}) < \varepsilon.$$

For any $i \in I$, there is $u_i : X \to \mathbb{R}$, such that $\lim_{p\to\infty} u_{pi}(x) = u_i(x)$ and u_i is continuous on X and, by Theorem 4.3.9 of [4], there is an $S_i \subset X_i$ such that $S_{pi} \to S_i$ and S_i is nonempty compact. Moreover, for any $m \geq p(\varepsilon)$,

$$\sup_{x \in X} \sum_{i=1}^{n} |u_{mi}(x) - u_i(x)| + \sum_{i=1}^{n} h_i(S_{mi}, S_i) \leq \varepsilon.$$

Since $\lambda_m = (u_{m1}, \ldots, u_{mn}; S_{m1}, \ldots, S_{mn}) \in \Lambda$, there is $x^m \in S_m = \prod_{i=1}^{n} S_{mi}$, such that for any $i \in I$, $u_{mi}(x_i^m, x_{\hat{i}}^m) = \max_{w_i \in S_{mi}} u_{mi}(w_i, x_{\hat{i}}^m)$.

Denote $S = \prod_{i=1}^{n} S_i$. Since $S_m \to S$,$x^m \in S_m$, by Lemma 1.1, there is $x \in S$ such that x is a cluster point of $\{x^m\}$ and, without loss of generality, we assume that $x^m \to x$.

Since

$$|u_{mi}(x_i^m, x_{\hat{i}}^m) - u_i(x_i, x_{\hat{i}})| \leq |u_{mi}(x_i^m, x_{\hat{i}}^m) - u_i(x_i^m, x_{\hat{i}}^m)| + |u_i(x_i^m, x_{\hat{i}}^m) - u_i(x_i, x_{\hat{i}})| \to 0,$$

then $u_{mi}(x_i^m, x_{\hat{i}}^m) \to u_i(x_i, x_{\hat{i}})$ and by Lemma 1.3, $\max_{w_i \in S_{mi}} u_{mi}(w_i, x_{\hat{i}}^m) \to \max_{w_i \in S_i} u_i(w_i, x_{\hat{i}})$. Hence $\forall i \in N$, $u_i(x_i, x_{\hat{i}}) = \max_{w_i \in S_i} u_i(w_i, x_{\hat{i}})$. Thus $\lambda = (u_1, \ldots, u_n; S_1, \ldots, S_n) \in \Lambda$. This proves that (Λ, ρ) is complete and the proof is finished.

Consider the model $M = \{\Lambda, X, F, R\}$: Λ is a complete metric space, X is a metric space, $\forall \lambda = (u_1, \ldots, u_n; S_1, \ldots, S_n) \in \Lambda, \forall x \in X$, define $F(\lambda, x) = S = \prod_{i=1}^{n} S_i$; $\forall \lambda \in \Lambda$, the correspondence $f(\lambda) = \{x \in X : x \in F(\lambda, x)\} = S$, which is continuous, and $f(\lambda)$ is nonempty compact for each $\lambda \in \Lambda$. $\forall \lambda \in \Lambda, \forall x \in f(\lambda) = S$, define

$$R(\lambda, x) = \sum_{i=1}^{n} [\max_{w_i \in S_i} u_i(w_i, x_{\hat{i}}) - u_i(x_i, x_{\hat{i}})].$$

$\forall \lambda \in \Lambda$, denote by $E(\lambda)$ the set of all Nash equilibria of the game λ, then $E(\lambda) \neq \emptyset$. It is easy to show that $R(\lambda, x) \geq 0$ and that $R(\lambda, x) = 0$ if and only if $x \in E(\lambda)$. □

Lemma 3.2 $R : \Lambda \times X \to \mathbb{R}_+$ *is continuous.*

Proof $\forall \lambda_m = (u_{m1}, \ldots, u_{mn}; S_{m1}, \ldots, S_{mn}) \in \Lambda, \lambda_m \to \lambda = (u_1, \ldots, u_n; S_1, \ldots, S_n)$, $\forall x^m \in f(\lambda_m), x^m \to x$, let us show that $R(\lambda_m, x^m) \to R(\lambda, x)$. In fact, $\forall i \in N$, we have shown in Lemma 3.1 that $u_{mi}(x_i^m, x_{\hat{i}}^m) \to u_i(x_i, x_{\hat{i}})$ and $\max_{w_i \in S_{mi}} u_{mi}(w_i, x_{\hat{i}}^m) \to \max_{w_i \in S_i} u_i(w_i, x_{\hat{i}})$. Hence $R(\lambda_m, x^m) \to R(\lambda, x)$. □

Now it follows that Theorems 2.1-2.3 hold for the model M above.

Remark 3.1 Note that there is no convexity conditions in the n-person noncooperative game problems while the existence of Nash equilibrium points is added in the definition of Λ.

3.2 Multiobjective optimization problem

Let X be a complete metric space and

$$\Lambda = \{\lambda = (\varphi, A) : \varphi = (\varphi_1, \ldots, \varphi_m) : X \to \mathbb{R}^m \text{ is continuous, } A \text{ is a nonempty compact subset of } X \text{ and } \sup_{x \in X} ||\varphi(x)|| < +\infty\}$$

For any $\lambda_1 = (\varphi^1, A_1), \lambda_2 = (\varphi^2, A_2) \in \Lambda$, define

$$\rho(\lambda_1, \lambda_2) = \sup_{x \in X} ||\varphi^1(x) - \varphi^2(x)|| + h(A_1, A_2)$$

where h is the Hausdorff metric on X. It is easy to show that

Lemma 3.3 (Λ, ρ) *is a complete metric space.*

Given any $\lambda = (\varphi, A)$, it defines a multiobjective optimization problem $\min_{x \in A} \varphi(x)$. Since φ is continuous and A is compact, by Corollary 1.2 (p.136) in [5], there must be an $x \in A$, such that for any $y \in A, \varphi(x) - \varphi(y) \notin int\mathbb{R}^m_+$ and x is called a weakly efficient solution of the multiobjective optimization problem. Denote by $E(\lambda)$ the set of all weakly efficient solutions of $\lambda \in \Lambda$.

Consider the model $M = \{\Lambda, X, F, R\}$, where Λ is a complete metric space, X is a metric space and $\forall \lambda = (\varphi, A) \in \Lambda, \forall x \in X$, define $F(\lambda, x) = A$. For any $\lambda \in \Lambda$, the correspondence $f(\lambda) = \{x \in X : x \in F(\lambda, x)\} = A$, which is continuous, and $f(\lambda)$ is nonempty compact. For any $\lambda \in \Lambda$ and any $x \in f(\lambda) = A$, define

$$R(\lambda, x) = \max_{y \in A} \min_{z \in Z} \langle z, \varphi(x) - \varphi(y) \rangle,$$

where $Z = \{z \in \mathbb{R}^m_+ : ||z|| = 1\}$. Note that since φ is continuous, $\langle z, \varphi(x) - \varphi(y) \rangle$ is continuous with respect to x, y and z, and Z is compact, then $\min_{z \in Z} \langle z, \varphi(x) - \varphi(y) \rangle$ is a continuous function with respect to (x, y). Moreover, since A is compact, $\max_{y \in A} \min_{z \in Z} \langle z, \varphi(x) - \varphi(y) \rangle$ must exist.

Lemma 3.4 (1) $\forall \lambda = (\varphi, A) \in \Lambda, \forall x \in f(\lambda) = A$, $R(\lambda, x) \geq 0$, *and* $R(\lambda, x) = 0$ *if and only if* $x \in E(\lambda)$.

(2) $R(\lambda, x)$ *is lower semicontinuous.*

Proof (1) Since $x \in A$, $R(\lambda, x) \geq \min_{z \in Z} \langle z, f(x) - f(x) \rangle = 0$. If $R(\lambda, x) = 0$, then $\forall y \in A, \min_{z \in Z} \langle z, \varphi(x) - \varphi(y) \rangle \leq 0$. If $x \notin E(\lambda)$, then there exists $y \in A$ such that $\varphi(x) - \varphi(y) \in int\mathbb{R}^m_+$ and thus for any $z \in Z$, $\langle z, \varphi(x) - \varphi(y) \rangle > 0$. By the compactness of X, there must be $\min_{z \in Z} \langle z, \varphi(x) - \varphi(y) \rangle > 0$, a contradiction. Hence $x \in E(\lambda)$.

Conversely, if $x \in E(\lambda)$, i.e., $\forall y \in A, \varphi(x) - \varphi(y) \notin int\mathbb{R}^m_+$. Let $I(y) = \{i : \varphi_i(x) - \varphi_i(y) \leq 0\}$, then $I(y) \neq \emptyset$. Take $i_0 \in I(y)$ and let $z^0 = (z_1, \ldots, z_m)$, where $z_{i_0} = 1, z_i = 0 (i \neq i_0)$. Then $z^0 \in Z$ and $\min_{z \in Z} \langle z, \varphi(x) - \varphi(y) \rangle \leq \langle z^0, \varphi(x) - \varphi(y) \rangle \leq 0$. Thus,

$$R(\lambda, x) = \max_{y \in A} \min_{z \in Z} \langle z, \varphi(x) - \varphi(y) \rangle \leq 0.$$

Note that we have shown that $R(\lambda, x) \geq 0$. Therefore, $R(\lambda, x) = 0$.

(2) We only need to show that $\forall \varepsilon > 0, \forall \lambda_n = (\varphi^n, A_n) \in \Lambda, \lambda_n \to \lambda = (\varphi, A), \forall x_n \in A_n, x_n \to x \in A$, then there exists a positive integer N, such that $\forall n \geq N$, $R(\lambda_n, x_n) > R(\lambda, x) - \varepsilon$.

First, there exists a $y \in A$ such that $\min_{z\in Z}\langle z, \varphi(x) - \varphi(y)\rangle = R(\lambda, x)$. For any $\varepsilon > 0$, since $\lambda_n \to \lambda$, we have $\varphi^n \to \varphi$, $A_n \to A$, which imply that there exists a positive integer N_1 such that $\forall n > N_1$, $\sup_{x\in X} ||\varphi^n(x) - \varphi(x)|| < \frac{\varepsilon}{4}$ and there is $y_n \in A_n$ such that $y_n \to y$. Then the continuity of φ implies that there is a positive integer N_2 such that $||\varphi(x_n) - \varphi(x)|| < \frac{\varepsilon}{4}$ and $||\varphi(y_n) - \varphi(y)|| < \frac{\varepsilon}{4}$ for all $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$||\varphi^n(x_n) - \varphi(x)|| \leq ||\varphi^n(x_n) - \varphi(x_n)|| + ||\varphi(x_n) - \varphi(x)|| < \frac{\varepsilon}{2},$$
$$||\varphi^n(y_n) - \varphi(y)|| \leq ||\varphi^n(y_n) - \varphi(y_n)|| + ||\varphi(y_n) - \varphi(y)|| < \frac{\varepsilon}{2}.$$

For any $z \in Z$, by Cauchy–Schwarz inequality,

$$\begin{aligned} &|\langle z, (\varphi^n(x_n) - \varphi^n(y_n)) - (\varphi(x) - \varphi(y))\rangle| \\ &\quad \leq |\langle z, (\varphi^n(x_n) - \varphi(x))\rangle| + |\langle z, (\varphi^n(y_n)) - \varphi(y))\rangle| \\ &\quad \leq ||z||||\varphi^n(x_n) - \varphi(x)|| + ||z||||\varphi^n(y_n) - \varphi(y)|| < \varepsilon. \end{aligned}$$

And thus

$$\langle z, \varphi^n(x_n) - \varphi^n(y_n)\rangle > \langle z, \varphi(x) - \varphi(y)\rangle - \varepsilon,$$

$$\min_{z\in Z}\langle z, \varphi^n(x_n) - \varphi^n(y_n)\rangle > \min_{z\in Z}\langle z, \varphi(x) - \varphi(y)\rangle - \varepsilon,$$

$$R(\lambda_n, x_n) \geq \min_{z\in Z}\langle z, \varphi^n(x_n) - \varphi^n(y_n)\rangle > \min_{z\in Z}\langle z, \varphi(x) - \varphi(y)\rangle - \varepsilon = R(\lambda, x) - \varepsilon. \qquad \square$$

Therefore, Theorems 2.1–2.4 hold for the model M above.

3.3 Fixed point problems

Let (X, d) be a bounded complete metric space and

$$\Lambda=\{\lambda=(\varphi, A) : \varphi : X\to K(X) \text{ is upper semicontinuous and } \exists x\in A \text{ such that } x \in \varphi(x)\}.$$

$\forall \lambda_1 = (\varphi_1, A_1), \lambda_2 = (\varphi_2, A_2) \in \Lambda$, define

$$\rho(\lambda_1, \lambda_2) = \sup_{x\in X} h(\varphi_1(x), \varphi_2(x)) + h(A_1, A_2),$$

where h is the Hausdorff metric on X.

Lemma 3.5 (Λ, ρ) *is a complete metric space.*

Proof Let $\{\lambda_m\}$ be any Cauchy sequence in Λ. Then for any $\varepsilon > 0$, there exists a positive integer $P(\varepsilon)$ such that for all $m, p \geq P(\varepsilon)$,

$$\rho(\lambda_m, \lambda_p) = \sup_{x\in X} h(\varphi_m(x), \varphi_p(x)) + h(A_m, A_p) < \varepsilon.$$

There exist $\varphi : X \to K(X)$ and $A \in K(X)$ such that $\varphi_p(x) \to \varphi(x)$, $A_p \to A$ and for all $m \geq P(\varepsilon)$,

$$\sup_{x\in X} h(\varphi_m(x), \varphi(x)) + h(A_m, A) \leq \varepsilon.$$

It is easy to check that φ is upper semicontinuous on X. Since $\lambda_m = (\varphi_m, A_m) \in \Lambda$, there are $x_m \in A_m$ such that $x_m \in \varphi_m(x_m)$, $m = 1, 2, \ldots$. It follows from $h(A_m, A) \to 0$ that there is an x which is a cluster of $\{x_m\}$ and, without loss of generality, we assume that $x_m \to x$.

Since φ is upper semicontinuous at x, there is an $m_0 \geq P(\varepsilon)$ such that for all $m \geq m_0$, $\varphi(x_m) \subset U(\varepsilon, \varphi(x)) = \{u \in X : d(u, y) < \varepsilon$, for some $y \in \varphi(x)\}$, and thus

$$x_m \in \varphi_m(x_m) \subset U(2\varepsilon, \varphi(x_m)) \subset U(3\varepsilon, \varphi(x)).$$

Letting $m \to \infty$, we obtain $d(x, \varphi(x)) \leq 3\varepsilon$. Since ε can be arbitrarily small, we have $d(x, \varphi(x)) = 0$ which implies that $x \in \varphi(x)$. Therefore, $\lambda = (\varphi, A) \in \Lambda$. Now we can conclude that (Λ, ρ) is a complete metric space.

Any given $\lambda = (\varphi, A) \in \Lambda$ determines a fixed point problem: finding $x \in A$ such that $x \in \varphi(x)$. Denote by $E(\lambda)$ the set of all fixed points of $\lambda \in \Lambda$. By the definition of Λ, we have $E(\lambda) \neq \emptyset$.

Consider the following model $M = \{\Lambda, X, F, R\}$, where Λ is a complete metric space and X is a metric space; $\forall \lambda = (\varphi, A) \in \Lambda$ and $\forall x \in X$, define $F(\lambda, x) = A$ and $\forall \lambda \in \Lambda$, the correspondence $f(\lambda) = \{x \in X : x \in F(\lambda, x)\} = A$, which is continuous and $f(\lambda)$ is nonempty compact set. $\forall \lambda \in \Lambda$, $\forall x \in f(\lambda) = A$, define

$$R(\lambda, x) = d(x, \varphi(x)) = \min_{y \in \varphi(x)} d(x, y),$$

where d is the distance function on X.

Clearly, $R(\lambda, x) \geq 0$. If $R(\lambda, x) = 0$, then $x \in \varphi(x)$, $x \in E(\lambda)$. Conversely, if $x \in E(\lambda)$, then $x \in A$, $x \in \varphi(x)$, $R(\lambda, x) = 0$. □

Lemma 3.6 *$R(\lambda, x)$ is lower semicontinuous.*

Proof It suffices to show that $\forall \varepsilon > 0$, $\forall \lambda_n = (\varphi_n, A_n) \in \Lambda$, $\lambda_n \to \lambda = (\varphi, A) \in \Lambda$, $\forall x_n \in A_n$, $x_n \to x \in A$, then there is a positive integer N, such that $\forall n \geq N$, $R(\lambda_n, x_n) > R(\lambda, x) - \varepsilon$. By Proposition 21 of [2](p.118), $d(x, \varphi(x)) = \min_{y \in \varphi(x)} d(x, y)$ is lower semicontinuous at x. Hence there is a positive integer N_1 such that $\forall n \geq N_1$, $\min_{y \in \varphi(x_n)} d(x_n, y) > \min_{y \in \varphi(x)} d(x, y) - \frac{\varepsilon}{2}$.

For each $n = 1, 2, \ldots$, there is a $y_n \in \varphi_n(x_n)$ such that $d(x_n, y_n) = \min_{y \in \varphi_n(x_n)} d(x_n, y)$. Since $h(\varphi_n(x_n), \varphi(x_n)) \to 0$, $y_n \in \varphi_n(x_n)$, then there is a positive integer N_2 such that $\forall n \geq N_2$, $\exists y'_n \in \varphi(x_n)$ such that $d(y_n, y'_n) < \frac{\varepsilon}{2}$.

Now letting $N = \max\{N_1, N_2\}$, then $\forall n \geq N$,

$$\begin{aligned} R(\lambda_n, x_n) &= \min_{y \in \varphi_n(x_n)} d(x_n, y) = d(x_n, y_n) \geq d(x_n, y'_n) - d(y'_n, y_n) \\ &\geq \min_{y \in \varphi(x_n)} d(x_n, y) - \frac{\varepsilon}{2} > \min_{y \in \varphi(x)} d(x, y) - \varepsilon = R(\lambda, x) - \varepsilon. \end{aligned}$$ □

Theorems 2.1–2.4 hold for the model M above.

Remark 3.2 Note that there is also no convexity conditions in the fixed point problems while the existence of fixed points was added in the definition of Λ, because convexity is only one sufficient condition for existence.

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